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ON HENSTOCK-STIELTJES INTEGRALS OF INTERVAL-VALUED FUNCTIONS ON TIME SCALES

Ju Han Yoon*

Abstract. In this paper we introduce the interval-valued Henstock-Stieltjes integral on time scales and investigate some properties of these integrals.

1. Introduction and preliminaries

The Henstock integral for real functions was first defined by Henstock [2] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Wiener and Feynman integrals. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [5] in 2006. In 2000, Congxin Wu and Zengtai Gong introduced the concept of the Henstock integral of interval-valued functions [6].

In this paper we introduce the concept of the Henstock-Stieltjes delta integral of interval-valued functions on time scales and investigate some properties of the integral.

A time scale T is a nonempty closed subset of real number $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in T$ we define the forward jump operator $\sigma(t) = \inf\{s \in T : s > t\}$ where $\inf \phi = \sup T$, while the backward jump operator $\rho(t) = \sup\{s \in T : s < t\}$ where $\sup \phi = \inf T$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ of $t \in T$ is defined by $\mu(t) = \sigma(t) - t$, whlie the backward graininess function $\nu(t)$ of $t \in T$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in T$ we denote the closed interval $[a, b]_T = \{t \in T : a \leq t \leq b\}$. $\delta = (\delta_L, \delta_R)$ is a Δ-gauge on $[a, b]_T$ if $\delta_L(t) > 0$ on $(a, b]_T$, $\delta_R(t) > 0$ on $(a, b]_T$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(t) \geq \mu(t)$ for each $t \in [a, b]_T$.

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A collection $P = \{([t_{i-1}, t_i], \xi_i) : 1 \le i \le n\}$ of tagged intervals is δ-fine Henstock partition of $[a,b]_T$ if $U_{i=1}^{n}[t_{i-1}, t_i] = [a,b]_T$, $[t_{i-1}, t_i]_T \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [t_{i-1}, t_i]_T$ for each $i = 1, 2, \cdots, n$.

DEFINITION 1.1 ([5]). *A function $f : [a,b] \longrightarrow \mathbb{R}$ is Henstock delta integrable (or H_Δ-integrable) on $[a,b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a Δ-gauge δ on $[a,b]$ such that*

$$\left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - A \right| < \epsilon$$

for every δ-fine Henstock partition $P = \{([t_{i-1}, t_i], \xi_i) : 1 \le i \le n\}$ of $[a,b]$. The number A is called the Henstock delta integral of f on $[a,b]$ and we write $A = (H_\Delta) \int_a^b f$.

DEFINITION 1.2. *Let $I_{\mathbb{R}} = \{I = [I^-, I^+]$ is the closed bounded interval on the real $\mathbb{R}\}$, where $I^- = \min\{x : x \in I\}$, $I^+ = \max\{x : x \in I\}$. For $A, B, C \in I_{\mathbb{R}}$, we define $A \le B$ iff $A^- \le B^-$ and $A^+ \le B^+$, $A + B = C$ iff $A^- + B^- = C^-$ and $A^+ + B^+ = C^+$, and $AB = \{ab : a \in A, b \in B\}$, where $(AB)^- = \min(A^-B^-, A^-B^+, A^+B^-, A^+B^+)$ and $(AB)^+ = \max(A^-B^-, A^-B^+, A^+B^-, A^+B^+)$. Define $d(A,B) = \max(|A^- - B^-|, |A^+ - B^+|)$ as the distance between A and B.*

DEFINITION 1.3 ([6]). *An interval-valued function $F : [a,b] \longrightarrow I_{\mathbb{R}}$ is Henstock integrable to $I_0 \in I_{\mathbb{R}}$ on $[a,b]$ if for every $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that*

$$d\left(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0 \right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i], \xi_i) : 1 \le i \le n\}$ of $[a,b]$ is a δ-fine Henstock partition of $[a,b]$. We write $(IH) \int_a^b F(x)dx = I_0$ and $F \in IH[a,b]$.

2. The interval-valued Henstock-Stieltjes delta integral on time scales

In this section, we will define the Henstock-Stieltjes integral of interval-valued function on time scales and investigate some properties of the integral.

DEFINITION 2.1. *Let α be an increasing function on $[a,b]_T$. An interval-valued function $F : [a,b]_T \longrightarrow I_{\mathbb{R}}$ is Henstock delta integrable* *with respect to α to $I_0 \in I_{\mathbb{R}}$ on $[a,b]_T$ if for every $\epsilon > 0$ there exists a Δ-gauge δ on $[a,b]_T$ such that*

$$d\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})), I_0\right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ is a δ-fine Henstock partition of $[a,b]_T$. We write $(IHS_\Delta)\int_a^b F(x)d\alpha = I_0$ and $F \in IHS_\Delta^\alpha[a,b]_T$.

Remark 2.2. *If $F(x) = F^-(x) = F^+(x)$ for all $x \in [a,b]_T$. It is clear that Definition 2.1 implies the real-valued Henstock-Steiltjes integral on $[a,b]_T$.*

Remark 2.3. *If $F \in IHS_\Delta^\alpha[a,b]_T$, then the integral is unique.*

Theorem 2.4. *An interval-valued function $F : [a,b]_T \longrightarrow I_{\mathbb{R}}$ is Henstock-Stieltjes delta integrable with respect to α on $[a,b]_T$ if and only if $F^-, F^+ \in HS_\Delta^\alpha[a,b]_T$ and*

$$(IHS_\Delta)\int_a^b F(x)d\alpha = \left[(HS_\Delta)\int_a^b F^-(x)d\alpha, (HS_\Delta)\int_a^b F^+(x)d\alpha\right],$$

where $F(x) = [F^-(x), F^+(x)]$.

Proof. Let $F \in IHS_\Delta^\alpha[a,b]_T$. Then there exists an interval $I_0 = [I_0^-, I_0^+]$ with the property that for each $\epsilon > 0$ there exists a Δ-gauge δ with respect to α on $[a,b]_T$ such that

$$d\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})), I_0\right) < \epsilon$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ is a δ-fine Henstock partition of $[a,b]_T$. Let $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ be a δ-fine Henstock partition of $[a,b]_T$. Since

$$\begin{aligned} &d\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})), I_0\right) \\ &= \max\left(\left|\sum_{i=1}^{n} F^-(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})) - I_0^-\right|, \left|\sum_{i=1}^{n} F^+(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})) - I_0^+\right|\right). \end{aligned}$$

Hence,

$$\left|\sum_{i=1}^{n} F^-(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})) - I_0^-\right| < \epsilon, \left|\sum_{i=1}^{n} F^+(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})) - I_0^+\right| < \epsilon.$$

Conversely, let $F^-, F^+ \in HS^{\alpha}_{\Delta}[a,b]_T$. then there exist $H_1, H_2 \in \mathbb{R}$ with the property that given Δ-gauge δ on $[a,b]_T$ such that

$$\left|\sum_{i=1}^{n} F^-(\xi_i)(\alpha(t_i)-\alpha(t_{i-1}))-H_1\right| < \epsilon, \left|\sum_{i=1}^{n} F^+(\xi_i)(\alpha(t_i)-\alpha(t_{i-1}))-H_2\right| < \epsilon.$$

whenever $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ is a δ-fine Henstock partition of $[a,b]_T$. Let $I_0 = [H_1, H_2]$. For any δ-fine Henstock partition $P = \{([t_{i-1}, t_i]_T, \xi_i) : 1 \le i \le n\}$ of $[a,b]_P$, we have

$$d\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(t_i)-\alpha(t_{i-1})), I_0\right) < \epsilon.$$

Hence $F : [a,b]_T \longrightarrow I_{\mathbb{R}}$ is Henstock-Stieltjes delta integrable with respect to α on $[a,b]_T$. $\square$

From Theorem 2.4 and the properties of Henstock-Stieltjes delta integral, we can easily obtain the following theorems.

THEOREM 2.5. *Let $F, G \in IHS^{\alpha}_{\Delta}[a,b]_T$ and $\beta, \gamma \in \mathbb{R}$. Then*

(1) $\beta F + \gamma G \in IHS^{\alpha}_{\Delta}[a,b]_T$ *and* $(IHS_{\Delta}[a,b]_T)\int_a^b (\beta F + \gamma G)\, d\alpha = \beta(IHS_{\Delta}[a,b]_T)\int_a^b F d\alpha + \gamma(IHS_{\Delta}[a,b]_T)\int_a^b G d\alpha$
(2) *If* $F(x) \le G(x)$ *a.e. in* $[a,b]_T$, *then* $(IHS_{\Delta}[a,b]_T)\int_a^b F d\alpha \le (IHS\Delta[a,b]_T)\int_a^b G d\alpha$

THEOREM 2.6. *Let $F \in IHS^{\alpha}_{\Delta}[a,c]_T$ and $F \in IHS^{\alpha}_{\Delta}[c,b]_T$. Then $F \in IHS^{\alpha}_{\Delta}[a,b]_T$ and*

$$(IHS_{\Delta})\int_a^b F d\alpha = (IHS_{\Delta})\int_a^c F d\alpha + (IHS_{\Delta})\int_c^b F d\alpha$$

THEOREM 2.7. *Let $F, G \in IHS^{\alpha}_{\Delta}[a,b]_T$ and $d(F,G)$ be a Henstock-Stieltjes delta integrable on $[a,b]_T$. Then*

$$d\left((IHS_{\Delta})\int_a^b F d\alpha, (IHS_{\Delta})\int_a^b G d\alpha\right) \le (H_{\Delta})\int_a^b d(F,G) d\alpha$$

Proof. By definition of distance, we have

$$
\begin{aligned}
& d\left((IHS_\Delta)\int_a^b F d\alpha, (IHS_\Delta)\int_a^b G d\alpha\right) \\
&= \max\left(\left|\left((IHS_\Delta)\int_a^b F d\alpha\right)^- - \left((IHS_\Delta)\int_a^b G d\alpha\right)^-\right|,\right. \\
&\qquad\qquad \left.\left|\left((IHS_\Delta)\int_a^b F d\alpha\right)^+ - \left((IHS_\Delta)\int_a^b G d\alpha\right)^+\right|\right) \\
&= \max\left(\left|(HS_\Delta)\int_a^b (F^- - G^-) d\alpha\right|, \left|(HS_\Delta)\int_a^b (F^+ - G^+) d\alpha\right|\right) \\
&\le \max\left((HS_\Delta)\int_a^b \left|F^- - G^-\right| d\alpha, (HS_\Delta)\int_a^b \left|F^+ - G^+\right| d\alpha\right) \\
&\le (HS_\Delta)\int_a^b d(F,G) d\alpha.
\end{aligned}
$$

□

3. The Henstock-Stieltjes delta integral of fuzzy number valued functions on time scales

Definition 3.1 ([1]). *Let $\widetilde{A} \in F(\mathbb{R})$ be a fuzzy subset on $\mathbb{R}$. If for any $\lambda \in [0,1]$, $A_\lambda = [A_\lambda^-, A_\lambda^+]$ and $A_1 \neq \emptyset$, where $A_\lambda = \{x : \widetilde{A}(x) \geq \lambda\}$, then $\widetilde{A}$ is called a fuzzy number.*

Let $\widetilde{\mathbb{R}}$ denote the set of all fuzzy numbers.

Definition 3.2 ([3]). *Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathbb{R}}$, we define $\widetilde{A} \leq \widetilde{B}$ iff $A_\lambda \leq B_\lambda$ for all $\lambda \in (0,1]$, $\widetilde{A} + \widetilde{B} = \widetilde{C}$ iff $A_\lambda + B_\lambda = C_\lambda$ for any $\lambda \in (0,1]$, $\widetilde{A} \cdot \widetilde{B} = \widetilde{D}$ iff $A_\lambda \cdot B_\lambda = D_\lambda$ for any $\lambda \in (0,1]$. For $D(\widetilde{A}, \widetilde{B}) = \sup_{\lambda \in [0,1]} d(A_\lambda, B_\lambda)$ is called the distance between $\widetilde{A}$, $\widetilde{B}$.*

Lemma 3.3 ([1]). *If a mapping $H : [0,1] \longrightarrow I_{\mathbb{R}}, \lambda \mapsto H(\lambda) = [m_\lambda, n_\lambda]$, satisfies $[m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}]$ when $\lambda_1 < \lambda_2$, then*

$$\widetilde{A} := \bigcup_{\lambda \in (0,1]} \lambda H(\lambda) \in \widetilde{\mathbb{R}}$$

and

$$A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n),$$

where $\lambda_n = [1 - 1/(n+1)]\lambda$.

DEFINITION 3.4. *Let* $\widetilde{F} : [a,b]_T \longrightarrow \widetilde{\mathbb{R}}$ *and let* α *be an increasing on* $[a,b]_T$. *If the interval-valued function* $F_\lambda(x) = \left[F_\lambda^-(x), F_\lambda^+(x)\right]$ *is Henstock-Stieltjes delta integrable with respect to* α *on* $[a,b]_T$ *for any* $\lambda \in (0,1]$, *then we say that* $\widetilde{F}(x)$ *is Henstock-Stieltjes delta integrable with respect to* α *on* $[a,b]_T$ *and the integral is defined by Henstock-Stieltjes delta integral as follow:*

$$\begin{aligned}(FHS_\Delta)\int_a^b \widetilde{F}(x)d\alpha &:= \bigcup_{\lambda\in(0,1]} \lambda (IHS_\Delta)\int_a^b F_\lambda(x)d\alpha \\ &= \bigcup_{\lambda\in(0,1]} \lambda \left[(HS_\Delta)\int_a^b F_\lambda^- d\alpha, (HS_\Delta)\int_a^b F_\lambda^+ d\alpha\right].\end{aligned}$$

We will write $\widetilde{F} \in FHS_\Delta^\alpha[a,b]_T$.

THEOREM 3.5. $\widetilde{F} \in FHS_\Delta^\alpha[a,b]_T$, *then* $(FHS_\Delta)\int_a^b \widetilde{F}(x)d\alpha \in \widetilde{\mathbb{R}}$ *and*

$$\left[(FHS_\Delta)\int_a^b \widetilde{F}(x)d\alpha\right]_\lambda = \bigcap_{n=1}^{\infty} (IHS_\Delta)\int_a^b F_{\lambda_n}(x)d\alpha,$$

where $\lambda_n = [1 - 1/(n+1)]\lambda$.

Proof. Let $H : (0,1] \longrightarrow I_{\mathbb{R}}$ be defined by

$$H(\lambda) = \left[(FH_\Delta)\int_a^b F_\lambda^-(x)d\alpha, (FH_\Delta)\int_a^b F_\lambda^+(x)d\alpha\right].$$

Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are increasing and decreasing on λ, respectively, therefore, when $0 < \lambda_1 \le \lambda_2 \le 1$ we have $F_{\lambda_1}^-(x) \le F_{\lambda_2}^-(x)$, $F_{\lambda_1}^+(x) \ge F_{\lambda_2}^+(x)$ on $[a,b]_T$. Thus from Theorem 2.5, we have

$$\begin{aligned}&\left[(HS_\Delta)\int_a^b F_{\lambda_1}^-(x)d\alpha, (HS_\Delta)\int_a^b F_{\lambda_1}^+(x)d\alpha\right] \\ &\qquad \supset \left[(HS_\Delta)\int_a^b F_{\lambda_2}^-(x)d\alpha, (HS_\Delta)\int_a^b F_{\lambda_2}^+(x)d\alpha\right].\end{aligned}$$

Using Theorem 2.5 and Lemma 3.3 we obtain

$$(FHS_\Delta)\int_a^b \widetilde{F}(x)d\alpha := \bigcup_{\lambda\in(0,1]} \lambda \left[(HS_\Delta)\int_a^b F_\lambda^-(x)d\alpha, (HS_\Delta)\int_a^b F_\lambda^+(x)d\alpha\right] \in \widetilde{\mathbb{R}}$$

and for all $\lambda \in (0,1]$,

$$\left[(FHS_\Delta)\int_a^b \widetilde{F}(x)d\alpha\right]_\lambda = \bigcap_{n=1}^{\infty}(IHS_\Delta)\int_a^b F_{\lambda_n}(x)d\alpha,$$

where $\lambda_n = [1 - 1/(n+1)]\lambda$. $\square$

Using Theorem 3.5 and the properties of (IHS) integral, we can obtain the properties of (FHS_Δ) integral. For examples, we get the linearity, monotonicity and interval additivity properties of (FHS_Δ) integral.

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Department of Mathematics Education
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: yoonjh@cbnu.ac.kr